

MEAN CURVATURE FLOWS IN ALMOST FUCHSIAN MANIFOLDS

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ABSTRACT. An almost Fuchsian manifold is a quasi-Fuchsian hyperbolic three-manifold that contains a closed incompressible minimal surface with principal curvatures everywhere in the range of $(-1, 1)$. In such a hyperbolic three-manifold, the minimal surface is unique and embedded, hence one can parametrize these three-manifolds by their minimal surfaces. We prove that any closed surface which is a graph over any fixed surface of small principal curvatures can be deformed into the minimal surface via the mean curvature flow. We also obtain an upper bound for the hyperbolic volume of the convex core of M^3 , as well as estimates on the Hausdorff dimension of the limit set.

1. INTRODUCTION

Quasi-Fuchsian three-manifold is an important class of complete hyperbolic 3-manifolds. In hyperbolic geometry, quasi-Fuchsian manifolds and their moduli space, the quasi-Fuchsian space, have been objects of extensive study in recent decades. Incompressible surfaces of small principal curvatures play an important role in low dimensional topology ([Rub05]). We denote M^3 in this paper an almost Fuchsian manifold: it is a quasi-Fuchsian manifold which contains a closed incompressible minimal surface Σ such that the principal curvatures of Σ are in the range of $(-1, 1)$.

The space of almost Fuchsian manifolds is a subspace of the quasi-Fuchsian space, of the same complex dimension $6g - 6$, where $g \geq 2$ is the genus of any closed incompressible surface in the manifold ([Uhl83]). Understanding the structures of the quasi-Fuchsian space is a mixture of understanding the hyperbolic geometry and topology of quasi-Fuchsian three-manifolds, the deformation of incompressible surfaces (induce injections between fundamental groups of the surface and the three-manifold), as well as the representation theory of Kleinian groups.

This notion of being almost Fuchsian (term coined in [KS07]) was first studied by Uhlenbeck ([Uhl83]), where she proved several key properties of almost Fuchsian manifolds that will be vital in this work: Σ is the only incompressible minimal surface in M^3 , and M^3 admits a foliation of parallel surfaces from Σ to both ends.

Our convention of the mean curvature is the sum of the principal curvatures, and we always assume the genus of any incompressible surface of M^3 is at least two.

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We also assume M^3 is not Fuchsian, or theorems are trivial. Surfaces we encounter in this paper will always be incompressible, unless otherwise stated.

Somewhat differently from [Wan08] and [HW09], where we used volume preserving mean curvature flow to address the issue of foliation of constant mean curvature surfaces, in this work, we use the usual mean curvature flow to deform a rather arbitrary closed surface S_0 to the minimal surface Σ . The special geometry of the ambient space (almost Fuchsian) allows us to apply the mean curvature flow to drag S_0 towards Σ rapidly and change the shapes of the evolving surfaces until the mean curvatures become zero.

Our main analytical tool is the mean curvature flow equation, which has the following form:

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} F(x, t) = -H(x, t)\nu(x, t) , \\ F(\cdot, 0) = F_0 , \end{cases}$$

where $H(x, t)$ is the mean curvature of the evolving surface $S(t)$, and all other terms will be made transparent in section two.

Our main result is that one can deform initial closed graphical surface S_0 over a fixed surface S with $|\lambda_j(S)| < 1$ to the unique minimal surface in an almost Fuchsian manifold:

Theorem 1.1. *Let M^3 be almost Fuchsian and S (S not necessarily the unique minimal surface of M^3) be any closed incompressible surface with principal curvatures in $(-1, 1)$ everywhere on S . Suppose a smooth closed surface S_0 in M^3 is a graph over S : there is a constant $c_0 > 0$ such that $\langle \mathbf{n}, \nu_0 \rangle \geq c_0$, where \mathbf{n} is the unit normal vector on S and ν_0 is the unit normal vector on S_0 . Then:*

- (i) *the mean curvature flow equation (1.1) with initial surface $S(0) = S_0$ has a long time solution;*
- (ii) *the evolving surfaces $\{S(t)\}_{t \in \mathbb{R}}$ stay smooth and remain as graphs over S for all time;*
- (iii) *$\{S(t)\}_{t \in \mathbb{R}}$ converge exponentially to the minimal surface Σ .*

We note that we do not require the principal curvatures of the initial surface S_0 to be small. In other words, the mean curvature flow (1.1) is very insensitive to the initial data. When M^3 is almost Fuchsian, there are abundant incompressible surfaces of small principal curvatures, for instance, every parallel surface from the minimal surface satisfies the principal curvature condition. This theorem is generalized to a larger class of quasi-Fuchsian manifolds (see Theorem 6.1) in §6.

It is very important to us that we are dealing with graphical surfaces: our estimates rely on the basic fact that the graph functions behave quite regularly in hyperbolic spaces under evolution equations (see for example [EH89], [Unt03]). Additional benefit of having graphical surfaces: these surfaces are naturally embedded, while it is generally very difficult to prove embeddedness from geometric measure theory.

Since the minimal surface is unique in an almost Fuchsian three-manifold, we can use minimal surfaces to parametrize the space of almost Fuchsian manifolds.

We obtain topological and geometric information about M^3 from data of Σ . In particular, the convex core of a quasi-Fuchsian three-manifold is a crucial part of the manifold. It is the smallest convex subset of a quasi-Fuchsian manifold that carries its fundamental group. From the point of view of hyperbolic geometry, the convex core contains all the geometrical information about the quasi-Fuchsian three-manifold itself (see for instance, [AC96], [Bro03]). As a direct application, when M^3 is almost Fuchsian, we obtain an explicit upper bound for the hyperbolic volume of $C(M^3)$, in terms of the maximum principal curvature on the minimal surface Σ :

Theorem 1.2. *If M^3 is almost Fuchsian, and let $\lambda_0 = \max_{x \in \Sigma} \{|\lambda(x)|\}$, then*

$$\begin{aligned} \text{vol}(C(M^3)) &\leq \mathcal{A}_{\text{hyp}} \left(\frac{\lambda_0}{1 - \lambda_0^2} + \frac{1}{2} \log \frac{1 + \lambda_0}{1 - \lambda_0} \right) \\ (1.2) \qquad &= \mathcal{A}_{\text{hyp}} \left(2\lambda_0 + \frac{4}{3}\lambda_0^3 + O(\lambda_0^5) \right), \end{aligned}$$

where $\mathcal{A}_{\text{hyp}} = 2\pi(2g - 2)$ is the hyperbolic area of any closed Riemann surface of genus $g > 1$.

We also obtain an upper bound for the Hausdorff dimension of the limit set Λ_Γ of M^3 , in terms of λ_0 as well:

Theorem 1.3. *If M^3 is almost Fuchsian, and let $\lambda_0 = \max_{x \in \Sigma} \{|\lambda(x)|\}$, then the Hausdorff dimension $D(\Lambda_\Gamma)$ of the limit set Λ_Γ for $M^3 = \mathbb{H}^3/\Gamma$ satisfies*

$$(1.3) \qquad D(\Lambda_\Gamma) < 1 + \lambda_0^2.$$

For λ_0 close to 0, Theorems 1.2 and 1.3 measure how close M is to being Fuchsian. In [Bro03], Brock showed the hyperbolic volume of the convex core is quasi-isometric to the Weil-Petersson distance between conformal boundaries of M^3 in Teichmüller space. We showed ([GHW10]) that the area functional of the minimal surface is a potential for the Weil-Petersson metric in Teichmüller space.

The differential equations of the evolution of hypersurfaces by their mean curvature have been studied extensively in various ambient Riemannian manifolds (see for instance [Bra78], [Hui84], [Hui86], [Hui87], and many others). Our study is of two-folds: the setting of quasi-Fuchsian hyperbolic three-manifolds provides effective barrier surfaces (Theorem 4.5) for the flow, and the convergence of the flow leads to the discovery of incompressible minimal surfaces in the three-manifolds.

Plan of the paper. We provide necessary background material in §2, especially the almost Fuchsian manifolds, the equidistant foliation, and the mean curvature flow. Section §3 is focused on the geometry of almost Fuchsian manifolds, where we prove the Theorem 1.2 (volume estimate) and Theorem 1.3 (Hausdorff dimension estimate). Special geometry of these three-manifolds plays important roles in our analysis of the mean curvature flow. We prove the Theorem 1.1 in next two sections: §4 (uniform bound for the second fundamental form and mean curvature), §5 (long time existence, convergence and uniqueness of the limit). We apply our technique

to more general cases in section §6, where we also show the mean convexity is preserved under the flow.

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2. PRELIMINARIES

In this section, we fix our notations, and introduce some preliminary facts that will be used in this paper.

2.1. Quasi-Fuchsian manifolds. For detailed reference on Kleinian groups and low dimensional topology, one can go to [Mar74] and [Thu82].

The universal cover of a hyperbolic three-manifold is \mathbb{H}^3 , and the deck transformations induce a representation of the fundamental group of the manifold in $Isom(\mathbb{H}^3) = PSL(2, \mathbb{C})$, the (orientation preserving) isometry group of \mathbb{H}^3 . A subgroup $\Gamma \subset PSL(2, \mathbb{C})$ is called a *Kleinian group* if Γ acts on \mathbb{H}^3 properly discontinuously. For any Kleinian group Γ , $\forall p \in \mathbb{H}^3$, the orbit set

$$\Gamma(p) = \{\gamma(p) \mid \gamma \in \Gamma\}$$

has accumulation points on the boundary $S_\infty^2 = \partial\mathbb{H}^3$, and these points are the *limit points* of Γ , and the closed set of all these points is called the *limit set* of Γ , denoted by Λ_Γ . In the case when Λ_Γ is contained in a circle $S^1 \subset S^2$, the quotient $M^3 = \mathbb{H}^3/\Gamma$ is called *Fuchsian*, and M^3 is isometric to a product space $S \times \mathbb{R}$. If the limit set Λ_Γ lies in a Jordan curve, $M^3 = \mathbb{H}^3/\Gamma$ is called **quasi-Fuchsian**, and it is topologically $S \times \mathbb{R}$, where S is a closed surface of genus g at least two. It is clear that a quasi-Fuchsian manifold is quasi-isometric to a Fuchsian manifold. The space of such three manifolds, the quasi-Fuchsian space of genus g surfaces, is a complex manifold of dimension of $6g - 6$, which has very complicated structures.

Finding minimal surfaces in negatively curved manifolds is a problem of fundamental importance. The basic results are due to Schoen-Yau ([SY79]) and Sacks-Uhlenbeck ([SU82]), and their results can be applied to the case of quasi-Fuchsian three-manifolds: any quasi-Fuchsian manifold contains at least one incompressible minimal surface. In the case of almost Fuchsian, the minimal surface is unique ([Uhl83]). On the other hand, there are many quasi-Fuchsian manifolds that admit many minimal surfaces ([Wan10]).

An essential problem in hyperbolic geometry and complex dynamics is to study the Hausdorff dimension $D(\Lambda_\Gamma)$ of the limit set Λ_Γ associated to M^3 . This problem also intimately related to understanding the lower spectrum theory of hyperbolic 3-manifolds ([Sul87, BC94]). In the case of Fuchsian manifolds, Λ_Γ is a round circle and $D(\Lambda_\Gamma) = 1$. When M^3 is quasi-Fuchsian but not Fuchsian, as throughout this paper, we have $1 < D(\Lambda_\Gamma) < 2$. There is a rich theory of quasiconformal mapping and its distortion in Hausdorff dimension, area and other quantities (see for instance [GV73, LV73]).

The following lemma is the well-known Hopf's maximum principle for tangential hypersurfaces in Riemannian geometry.

Lemma 2.1 ([Hop89]). *Let S_1 and S_2 be two hypersurfaces in a Riemannian manifold which intersect at a common point tangentially. If S_2 lies in positive side of S_1 around the common point, then $H_1 \leq H_2$, where H_i is the mean curvature of S_i at the common point for $i = 1, 2$.*

2.2. Almost Fuchsian manifolds. Throughout the paper, M^3 is an almost Fuchsian three-manifold: the principal curvatures of the minimal surface Σ are in the range of $(-1, 1)$.

The induced metric on an incompressible surface S is $g_{ij}(x) = e^{2v(x)}\delta_{ij}$, where $v(x)$ is a smooth function on S , and while the second fundamental form $A(x) = [h_{ij}]_{2 \times 2}$, here h_{ij} is given by, for $1 \leq i, j \leq 2$,

$$h_{ij} = \langle \bar{\nabla}_{e_i} \nu, e_j \rangle = -\langle \bar{\nabla}_\nu e_i, e_j \rangle ,$$

where $\{e_1, e_2\}$ is a basis on S , and ν is the unit normal field on S , and $\bar{\nabla}$ is the Levi-Civita connection of $(M^3, \bar{g}_{\alpha\beta})$.

We add a bar on top for each quantity or operator with respect to $(M^3, \bar{g}_{\alpha\beta})$.

Let $\lambda_1(x)$ and $\lambda_2(x)$ be the eigenvalues of $A(x)$. They are the principal curvatures of S , and we denote $H(x) = \lambda_1(x) + \lambda_2(x)$ as the mean curvature function of S .

Let $S(r)$ be the family of equidistant surfaces with respect to S , i.e.

$$S(r) = \{\exp_x(r\nu) \mid x \in S\} , \quad r \in (-\varepsilon, \varepsilon) .$$

The induced metric on $S(r)$ is denoted by $g(x, r) = g_{ij}(x, r)$, and the second fundamental form is denoted by $A(x, r) = [h_{ij}(x, r)]_{1 \leq i, j \leq 2}$. The mean curvature on $S(r)$ is thus given by $H(x, r) = g^{ij}(x, r)h_{ij}(x, r)$.

Lemma 2.2 ([Uhl83], [HW09]). *The induced metric $g(x, r)$ on $S(r)$ has the form*

$$(2.1) \quad g(x, r) = e^{2v(x)}[\cosh(r)\mathbb{I} + \sinh(r)e^{-2v(x)}A(x)]^2 ,$$

where $r \in (-\varepsilon, \varepsilon)$.

The principal curvatures of $S(r)$ are given by

$$(2.2) \quad \mu_j(x, r) = \frac{\tanh(r) + \lambda_j(x)}{1 + \lambda_j(x)\tanh(r)} , \quad j = 1, 2 .$$

and mean curvature is

$$(2.3) \quad H(x, r) = \frac{2(1 + \lambda_1\lambda_2)\tanh(r) + (\lambda_1 + \lambda_2)(1 + \tanh^2(r))}{1 + (\lambda_1 + \lambda_2)\tanh(r) + \lambda_1\lambda_2\tanh^2(r)} .$$

Clearly, when $|\lambda_j(x)| < 1$ for $j = 1, 2$, the metrics $g(x, r)$ are of no singularity for all $r \in \mathbb{R}$ and therefore $\{S(r)\}_{r \in \mathbb{R}}$ forms a foliation of surfaces parallel to S , called the *equidistant foliation* or the *normal flow*. In what follows, we have $|\lambda_j(S)| < 1$ for $j = 1, 2$.

Since M^3 is almost Fuchsian, then we have $|\lambda_j(x)| < 1$ for $j = 1, 2$ and $x \in \Sigma$. The equidistant foliation from the minimal surface Σ is then denoted by $\{\Sigma(r)\}_{r \in \mathbb{R}}$. Each fiber $\Sigma(r)$ satisfies the principal curvatures lie in $(-1, 1)$. We may assume the

reference surface $S = \Sigma$ to simplify the exposition in the first part of the proof of Theorem 4.5.

The existence of equidistant foliation is a remarkable property of almost Fuchsian manifolds. Recently, very interesting applications of almost Fuchsian manifolds have been explored in the content of mathematical physics, see for example, [KS07], [KS08] and others.

2.3. Mean curvature flow. Let $F_0 : S \rightarrow M^3 = S \times \mathbb{R}$ be the immersion of an incompressible surface S in M^3 . Without loss of generality, we assume that $S_0 = F_0(S)$ is contained in the positive side of the minimal surface Σ , and is a graph over Σ with respect to \mathbf{n} , i.e., $\langle \mathbf{n}, \nu_0 \rangle \geq c_0 > 0$, here \mathbf{n} is the unit normal vector on Σ , ν_0 is the unit normal vector on S_0 and c_0 is a constant depending only on S_0 .

We consider a family of immersions of surfaces in M^3 ,

$$F : S \times [0, T) \rightarrow M^3, \quad 0 \leq T \leq \infty$$

with $F(\cdot, 0) = F_0$. For each $t \in [0, T)$, $S(t) = \{F(x, t) \in M^3 \mid x \in S\}$ is the evolving surface at time t , and $H(x, t)$ its mean curvature.

The mean curvature flow equation is given by, as in (1.1):

$$\begin{cases} \frac{\partial}{\partial t} F(x, t) = -H(x, t)\nu(x, t), \\ F(\cdot, 0) = F_0, \end{cases}$$

Here $\nu(x, t)$ is the normal vector on $S(t)$ with $-\nu$ points to the reference surface S .

This is a parabolic equation, and Huisken proved the short-time existence of the solutions to (1.1), and initial compact surface quickly develops singularities along the flow, moreover, he showed the blow-up of the norm of the second fundamental forms if the singularity occurs in finite time.

Theorem 2.3 ([Hui84]). *If the initial surface S_0 is smooth, then the equation (1.1) has a smooth solution on some maximal open time interval $0 \leq t < T$, where $0 < T \leq \infty$. If $T < \infty$, then $|A|_{\max}(t) \equiv \max_{x \in S} |A|(x, t) \rightarrow \infty$ as $t \rightarrow T$.*

3. GEOMETRY OF ALMOST FUCHSIAN MANIFOLDS

In this section, we wish to obtain information about the almost Fuchsian manifold M^3 via its unique minimal surface Σ . We derive several geometrical properties on the equidistant foliation $\{\Sigma(r)\}_{r \in \mathbb{R}}$ in §3.1, and in §3.2, we obtain explicit upper bounds for the hyperbolic volume of the convex core of M^3 . Proposition 3.4 is particularly useful both in the proof of Theorem 4.5 and Theorem 1.2.

3.1. Some estimates on $\{\Sigma(r)\}_{r \in \mathbb{R}}$. We record principal curvatures of the minimal surface Σ by $\pm\lambda$ and $-1 < \lambda < 1$, and $|S|$ be the area for any incompressible surface S (with respect to the induced metric), and $\mathcal{A}_{\text{hyp}} = 2\pi(2g - 2)$ the hyperbolic area of the surface S .

We start with a well-known estimate which implies the area of the minimal surface under the induced metric from ambient space is comparable to that of the

hyperbolic area, with universal constants. We only include a proof for the sake of completeness.

Proposition 3.1. $\mathcal{A}_{\text{hyp}}/2 < |\Sigma| < \mathcal{A}_{\text{hyp}}$.

Proof. We apply the Gauss equation:

$$K(\Sigma) = -1 + \det(A),$$

where $K(\Sigma)$ is the Gaussian curvature of Σ and A is the second fundamental form of Σ .

Thus we have

$$-K(\Sigma) = 1 - \det(A) = 1 + \lambda^2.$$

We integrate this on Σ , applying the Gauss-Bonnet theorem since Σ is incompressible, to find

$$|\Sigma| < |\Sigma| + \int_{\Sigma} \lambda^2 = \mathcal{A}_{\text{hyp}} < 2|\Sigma|.$$

□

We want to estimate the area of each parallel surface in the equidistant foliation $\{\Sigma(r)\}_{r \in \mathbb{R}}$:

Proposition 3.2. *For all $-\infty < r < +\infty$, we have*

$$\frac{\mathcal{A}_{\text{hyp}}}{2} < |\Sigma| \leq |\Sigma(r)| \leq |\Sigma| \cosh^2(r) < \mathcal{A}_{\text{hyp}} \cosh^2(r)$$

Proof. The area element of $\Sigma(r)$ is given by

$$(3.1) \quad d\mu(r) = (\cosh^2(r) - \lambda^2(x) \sinh^2(r)) d\mu,$$

where $d\mu$ is the area element for the minimal surface Σ .

We can now compute the surface area:

$$\begin{aligned} |\Sigma(r)| &= \int_{\Sigma} (\cosh^2(r) - \lambda^2(x) \sinh^2(r)) d\mu \\ &= \cosh^2(r) |\Sigma| - \sinh^2(r) \int_{\Sigma} \lambda^2(x) d\mu \\ &= \cosh^2(r) |\Sigma| - \sinh^2(r) (\mathcal{A}_{\text{hyp}} - |\Sigma|) \\ &= |\Sigma| (\cosh^2(r) + \sinh^2(r)) - \sinh^2(r) \mathcal{A}_{\text{hyp}} \\ (3.2) \quad &= |\Sigma| + \sinh^2(r) (2|\Sigma| - \mathcal{A}_{\text{hyp}}), \end{aligned}$$

Here we used the identity

$$(3.3) \quad \int_{\Sigma} \lambda^2 = \mathcal{A}_{\text{hyp}} - |\Sigma|.$$

The estimates then follows from the Proposition 5.1 and (5.2). □

We also need the estimates on the mean curvatures of $\Sigma(r)$:

Proposition 3.3. $|H(\Sigma(r))| \leq 2|\tanh(r)| < 2$.

Proof. We only prove the part when $r > 0$. The mean curvature of the surface $\Sigma(r)$ is given by the formula (2.3):

$$H(x, r) = \frac{2(1 - \lambda^2(x)) \tanh(r)}{1 - \lambda^2(x) \tanh^2(r)}, \quad \forall x \in \Sigma.$$

An easy calculation shows

$$H(x, r) - 2 \tanh(r) = \frac{2\lambda^2 \tanh(r)(\tanh^2(r) - 1)}{1 - \lambda^2 \tanh^2(r)} \leq 0.$$

□

3.2. Upper bound for the convex core volume. We obtain an upper bound for the hyperbolic volume of the convex core $C(M^3)$ in this subsection, in terms of the maximum, λ_0 , of the principal curvature function on the minimal surface Σ .

We recall from formula (2.2), that the principal curvatures of the surface $\Sigma(r)$, $\mu_1(x, r)$ and $\mu_2(x, r)$ are increasing functions of r for any fixed $x \in \Sigma$, and they approach ± 2 as $r \rightarrow \pm\infty$. We also have $\mu_1(x, r) \leq \mu_2(x, r)$ for fixed r and x .

We are particularly interested in two critical cases: the values of r when $\mu_1(x, r) = 0$ and $\mu_2(x, r) = 0$. Elementary algebra shows:

Proposition 3.4. *If we denote*

$$r_0 = \frac{1}{2} \log \frac{1 + \lambda_0}{1 - \lambda_0},$$

where $\lambda_0 = \max_{x \in S} \{\lambda(x)\}$, then r_0 is the least value of r such that $\mu_1(r, x) > 0$ for all $r > r_0$, and $-r_0$ is the largest value for $\mu_2(r, x) < 0$ such that $\mu_2(r, x) < 0$ for all $r < -r_0$.

This Proposition tells us when the parallel surfaces in the equidistant foliation $\{\Sigma(r)\}_{r \in \mathbb{R}}$ become convex, hence by the definition of the convex core, provides an upper bound for the size of the convex core.

We denote the region of M^3 bounded between the surfaces $\Sigma(-r_0)$ and $\Sigma(r_0)$ by $M(r_0)$, and then the convex core $C(M^3)$, is contained in $M(r_0)$. Since $\{\Sigma(r)\}_{r \in \mathbb{R}}$ foliates M^3 , we can compute the hyperbolic volume of the region $M(r_0)$ by

$$\begin{aligned} \text{vol}(M(r_0)) &= \int_{-r_0}^{r_0} |\Sigma(r)| dr \\ &= 2r_0 |\Sigma| + (2|\Sigma| - \mathcal{A}_{\text{hyp}}) \int_{-r_0}^{r_0} \sinh^2(r) dr \\ &= 2r_0 |\Sigma| + (2|\Sigma| - \mathcal{A}_{\text{hyp}}) \left(\frac{1}{2} \sinh(2r_0) - r_0 \right) \\ (3.4) \quad &= |\Sigma| \sinh(2r_0) - \mathcal{A}_{\text{hyp}} \left(\frac{1}{2} \sinh(2r_0) - r_0 \right) \end{aligned}$$

Applying the Proposition 3.2, we obtain the following:

Theorem 3.5. *The hyperbolic volume of $C(M^3)$ is bounded by:*

$$\text{vol}(C(M^3)) \leq \mathcal{A}_{\text{hyp}}(\cosh r_0 \sinh r_0 + r_0)$$

$$(3.5) \quad = \mathcal{A}_{\text{hyp}} \left(\frac{\lambda_0}{1 - \lambda_0^2} + \frac{1}{2} \log \frac{1 + \lambda_0}{1 - \lambda_0} \right).$$

This estimate in the Theorem 3.5 can also be obtained via an application of the Cauchy-Schwarz inequality and the Propositions 3.2 and 3.3.

When r_0 , or equivalently, $\lambda(x) = 0$ for all $x \in \Sigma$, this is the case of M being Fuchsian, in which case, the hyperbolic volume of $C(M^3)$ is zero. We want to measure how the volumes vary for small λ_0 . From Taylor series expansion we have

Corollary 3.6. *For small λ_0 , we have the following expansion:*

$$\text{vol}(C(M^3)) \leq \mathcal{A}_{\text{hyp}} \left(2\lambda_0 + \frac{4}{3}\lambda_0^3 + O(\lambda_0^5) \right).$$

3.3. Hausdorff dimension of the limit set. We denote $C_1(M^3)$ the hyperbolic radius one neighborhood of the convex core $C(M^3)$ in M^3 . An easy calculation from (3.4) gives us

$$\text{vol}(C_1(M^3)) \leq \text{vol}(M(r_0 + 1)) \leq \mathcal{A}_{\text{hyp}} \left(\frac{1}{2} \sinh(2r_0 + 2) - r_0 - 1 \right),$$

where $r_0 = \frac{1}{2} \log \frac{1 + \lambda_0}{1 - \lambda_0}$. Therefore we have

$$(3.6) \quad \text{vol}(C_1(M^3)) \leq \mathcal{A}_{\text{hyp}} \left(\frac{1}{2} \sinh(\log \frac{1 + \lambda_0}{1 - \lambda_0} + 2) - \frac{1}{2} \log \frac{1 + \lambda_0}{1 - \lambda_0} - 1 \right).$$

Since quasi-Fuchsian manifolds are geometrically finite and infinite volume, and we assume M^3 is not Fuchsian, a direct application of the main theorem from Burger-Canary ([BC94]) gives:

Proposition 3.7. *Let M^3 be almost Fuchsian, and $\Lambda_0(M^3)$ be the bottom of the L^2 -spectrum of $-\Delta$ on M^3 , and $D(\Lambda_\Gamma)$ be the Hausdorff dimension of the limit set Λ_Γ of M^3 . Then we have*

(i)

$$(3.7) \quad \Lambda_0(M^3) \geq \frac{K_3}{\mathcal{A}_{\text{hyp}}^2 \left(\frac{1}{2} \sinh(\log \frac{1 + \lambda_0}{1 - \lambda_0} + 2) - \frac{1}{2} \log \frac{1 + \lambda_0}{1 - \lambda_0} - 1 \right)^2}.$$

(ii)

$$(3.8) \quad D(\Lambda_\Gamma) \leq 2 - \frac{K_3}{\mathcal{A}_{\text{hyp}}^2 \left(\frac{1}{2} \sinh(\log \frac{1 + \lambda_0}{1 - \lambda_0} + 2) - \frac{1}{2} \log \frac{1 + \lambda_0}{1 - \lambda_0} - 1 \right)^2}.$$

Here K_3 can be chosen such that $K_3 > 10^{-11}$.

We note that while the volume estimate of the convex core of M^3 (Theorem 3.5) is effective for small maximal principal curvature λ_0 of the minimal surface Σ , above estimates on $\Lambda_0(M^3)$ and $D(\Lambda_\Gamma)$ are not as effective. To obtain a better estimate, we consider the limit set Λ_Γ of M^3 as a k -quasircle (an image of a circle under a k -quasiconformal mapping).

A k -quasiconformal mapping f is a homomorphism of planar domains, locally in the Sobolev class W_2^1 such that its Beltrami coefficient $\mu_f = \frac{\bar{\partial}f}{\partial f}$ has bounded L^∞ bound: $\|\mu_f\| \leq k < 1$. One can visualize f infinitesimally maps a round

circle to an ellipse with a bounded dilatation $K = \frac{1+k}{1-k}$, where $k \in [0, 1)$. Clearly, the mapping f is conformal when $k = 0$. This is an important generalization of conformal maps. The study of quasiconformal mappings is a major breakthrough of geometric function theory behind Teichmüller's insight and Ahlfors-Bers' revival of Teichmüller theory.

Proof of Theorem 1.3. Let $\Sigma \subset M^3$ be the incompressible minimal surface with principal curvatures in $(-1, 1)$. We know that the normal bundle over Σ is trivial, i.e., the geodesics perpendicular to Σ are disjoint from each other. Therefore, any point $p \in M$ can be represented by $p = (x, r)$, here x is the projection of p to Σ along the geodesic which passes through p and is perpendicular to Σ , and r is the (signed) distance between p and x .

Now we can construct a Fuchsian 3-manifold $N = \Sigma \times \mathbb{R}$ as follows. Suppose that the induced metric on $\Sigma \subset M^3$ is given by $g(x) = e^{2v(x)}I$, here $v(x)$ is a smooth function defined on Σ and I is the 2×2 identity matrix. The metric $\bar{\rho}$ of N is given by

$$\bar{\rho}(x, r) = \begin{pmatrix} \rho(x, r) & 0 \\ 0 & 1 \end{pmatrix},$$

here $\rho(x, r) = e^{2v(x)} \cosh r I$. By the construction, it's easy to know that the surface $\Sigma \times \{0\}$ is totally geodesic. Similarly, any point $q \in N$ can be represented by $q = (y, s)$, here y is the projection of q to $\Sigma \times \{0\}$ and s is the distance between q and y .

Then we may define a map $\varphi : N \rightarrow M^3$ by $\varphi(x, r) = (x, r)$ for $(x, r) \in N$. By the result in [Uhl83, p. 162], the map φ is a quasi-isometry. Lift φ to the map $\tilde{\varphi} : \mathbb{H}^3 \rightarrow \mathbb{H}^3$, then $\tilde{\varphi}$ is also a quasi-isometry. By the results in [Geh62, Theorem 9], [Mos68, Theorem 12.1], [Thu82, Corollary 5.9.6] and [MT98, Theorem 3.22], $\tilde{\varphi}$ can be extended to an automorphism

$$(3.9) \quad \tilde{\varphi} : \mathbb{H}^3 \cup \widehat{\mathbb{C}} \rightarrow \mathbb{H}^3 \cup \widehat{\mathbb{C}}$$

such that the restriction $\tilde{\varphi}|_{\widehat{\mathbb{C}}} =: f$ is a quasiconformal mapping. In particular, f maps S_{\pm}^2 to $\Omega_{\pm}(\Gamma)$, here $S_{\pm}^2 = S_{\infty}^2 \setminus S^1$ are hemispheres such that $\partial S_+^2 = S^1 = \partial S_-^2$, and $f(S^1) = \Lambda_{\Gamma}$ respectively.

We claim that $f|_{S_+^2} : S_+^2 \rightarrow \Omega_+(\Gamma)$ is a k -quasiconformal mapping, with the dilatation $K = \frac{1+k}{1-k}$, and

$$K < \frac{1 + \lambda_0}{1 - \lambda_0}.$$

Let Π be the lift of the totally geodesic minimal surface $\Sigma \times \{0\} \subset N$, and let $\widehat{\Sigma}$ be the lift of the surface $\Sigma \subset M^3$. Recall that the identity map between Π and $\widehat{\Sigma}$ is an isometry, and we can define hyperbolic Gauss maps $G'_+ : \Pi \rightarrow S_+^2$ and $G''_+ : \widehat{\Sigma} \rightarrow \Omega_+(\Gamma)$ ([Eps86]) such that we have the following commutative diagram

$$\begin{array}{ccc} S_+^2 & \xrightarrow{f} & \Omega_+(\Gamma) \\ G'_+ \uparrow & & \uparrow G''_+ \\ \Pi & \xrightarrow{\text{id}} & \widehat{\Sigma} \end{array}$$

Since G'_+ is a conformal mapping and id is an isometry, we know that $G''_+ \circ f$ is also a conformal mapping. By Proposition 5.1 and Corollary 5.3 in [Eps86], $(G''_+)^{-1}$ is a k -quasiconformal mapping, so is f .

In particular, $\Lambda_\Gamma = f(S^1)$ is a k -quasicircle. Recently, Smirnov ([Smi09]) proved Astala's conjecture ([Ast94]): the Hausdorff dimension of a k -quasicircle is at most $1 + k^2$. Now

$$k = \frac{K-1}{K+1} < \lambda_0.$$

This proves the Theorem 1.3. □

4. PROOF OF THEOREM 1.1: UNIFORM BOUNDS

The next two sections are set up to prove the Theorem 1.1. The strategy is standard, though technical: we establish the uniform bounds of the square norm of the second fundamental form, $|A|^2$, and the square of the mean curvature, H^2 , as well as their derivatives; we bound the evolving surfaces $S(t)$ in a compact region of M^3 by a height function estimate; these bounds enables us to extend the solution of (1.1) beyond its maximal finite time interval; we show exponential convergence to prove the uniqueness of the limiting surface.

4.1. Some evolution equations. In this subsection, we collect and derive a number of evolution equations of some quantities and operators on $S(t)$, $t \in [0, T)$, which are involved in our calculations. These quantities and operators are:

- (i) the induced metric of $S(t)$: $g(t) = \{g_{ij}(t)\}$;
- (ii) the second fundamental form of $S(t)$: $A(\cdot, t) = \{h_{ij}(\cdot, t)\}$;
- (iii) the mean curvature of $S(t)$ with respect to the normal vector pointing to Σ : $H(\cdot, t) = g^{ij}h_{ij}$;
- (iv) the square norm of the second fundamental form of $S(t)$:

$$|A(\cdot, t)|^2 = g^{ij}g^{kl}h_{ik}h_{jl} ;$$

- (v) the covariant derivative of $S(t)$ is denoted by ∇ and the Laplacian on $S(t)$ is given by $\Delta = g^{ij}\nabla_i\nabla_j$.

We start with some standard evolution equations:

Lemma 4.1. ([Hui86]) *The evolution equations of the induced metric g_{ij} , the normal vector field ν , and the area element $d\mu$ are given by*

$$(4.1) \quad \frac{\partial}{\partial t} g_{ij} = -2Hh_{ij} ,$$

$$(4.2) \quad \frac{\partial}{\partial t} \nu = \nabla H ,$$

$$(4.3) \quad \frac{\partial}{\partial t} d\mu = -H^2 d\mu$$

It is clear from (4.3) that the mean curvature flow decreases the areas of the evolving surfaces.

We also need the following evolution equations for the mean curvature $H(\cdot, t)$, and the square norm of the second fundamental form $|A(\cdot, t)|^2$:

Lemma 4.2.

$$(4.4) \quad \left(\frac{\partial}{\partial t} - \Delta \right) H = H(|A|^2 - 2) ,$$

$$(4.5) \quad \left(\frac{\partial}{\partial t} - \Delta \right) |A|^2 = -2|\nabla A|^2 + 2|A|^2(|A|^2 - 2) + 4(2|A|^2 - H^2) .$$

Proof. These equations are deduced for general Riemannian manifolds in [Hui86]. In our case of hyperbolic three-manifold, the ambient space M^3 has constant sectional curvature -1 , and the Ricci curvature $Ric(\nu, \nu) = -2$ for any unit normal vector ν .

The lemma then follows from combining these explicit curvature conditions and curvature equations $\bar{R}_{3i3j} = -g_{ij}$, as well as the well-known Simons' identity (see [Sim68] or [SSY75]), satisfied by the second fundamental form h_{ij} :

$$\Delta h_{ij} = \nabla_i \nabla_j H - (|A|^2 - 2)h_{ij} + Hh_{il}h_{lj} + Hg_{ij} .$$

□

Our estimates will also involve the *height function* $u(x, t)$ and the *gradient function* $\Theta(x, t)$ on evolving surfaces $S(t)$:

$$(4.6) \quad u(x, t) = \ell(F(x, t))$$

$$(4.7) \quad \Theta(x, t) = \langle \nu(x, t), \mathbf{n} \rangle .$$

Here T_{\max} is the right endpoint of the maximal closed time interval on which the solution to exists, and $\ell(p) = \pm \text{dist}(p, S)$ for all $p \in M^3$, the distance to the reference surface S .

We always have $\Theta(x, t) \in [0, 1]$. It is clear that the surface $S(t)$ is a graph over S if $\Theta > 0$ on $S(t)$.

Our main assumption on the initial surface S_0 is that $\Theta(x, 0) \geq c_0 > 0$. Geometrically, one can view the initial surface has bounded geometry.

Lemma 4.3. ([Bar84], [EH91]) *The evolution equations of u and Θ have the following forms*

$$(4.8) \quad \frac{\partial}{\partial t} u = -H\Theta ,$$

$$(4.9) \quad \left(\frac{\partial}{\partial t} - \Delta \right) \Theta = (|A|^2 - 2)\Theta + \mathbf{n}(H_{\mathbf{n}}) - H\langle \bar{\nabla}_{\nu} \mathbf{n}, \nu \rangle .$$

Here $\bar{\nabla}$ is the gradient operator with respect to the hyperbolic metric, div is the divergence on $S(t)$, and $\mathbf{n}(H_{\mathbf{n}})$ is the variation of mean curvature function of $S(t)$ under the deformation vector field \mathbf{n} .

4.2. Positivity of $\Theta(\cdot, t)$. In this subsection, we establish the fact that evolving surfaces $S(t)$ remain as the graphs over S , provided the initial surface S_0 is a graph over S . This step will be important in the proof of Theorem 4.5 where we show all evolving surfaces stay in a compact region along the flow.

Lemma 4.4. *If $\Theta(\cdot, 0) \geq c_0 > 0$, where c_0 is any positive constant only depending on the initial surface, then $\Theta(\cdot, t) > 0$ for $t \in [0, T)$.*

Proof. Let

$$\Theta_{\min}(t) = \min_{x \in S} \Theta(x, t) .$$

We estimate the terms in the evolution equation (4.9) of Θ , starting with the expression $\mathbf{n}(H_{\mathbf{n}})$, from ([Bar84, Eq. (2.10)]):

$$|\mathbf{n}(H_{\mathbf{n}})| \leq c_1(\Theta^3 + \Theta^2|A|) ,$$

for some $c_1 > 0$.

We also have the following estimate from ([Eck03, Page 187])

$$|\langle \bar{\nabla}_{\nu} \mathbf{n}, \nu \rangle| \leq c_2 \Theta^2 ,$$

where $c_2 = \|\bar{\nabla} \mathbf{n}\| > 0$.

Collecting these estimates, and we derive from the equation (4.9):

$$\begin{aligned} \frac{d}{dt} \Theta_{\min} &\geq (|A|^2 - 2)\Theta_{\min} - c_1(\Theta_{\min}^3 + |A|\Theta_{\min}^2) - c_2|H|\Theta_{\min}^2 \\ &\geq ((|A|^2 - 2) - c_1(1 + |A|) - c_2\sqrt{2}|A|)\Theta_{\min} \\ &= (|A|^2 - (c_1 + c_2\sqrt{2})|A| - (2 + c_1))\Theta_{\min} \end{aligned}$$

Since $\Theta_{\min}(0) \geq c_0 > 0$, then above forces $\Theta_{\min} \geq 0$ and hence

$$\frac{d}{dt} \Theta_{\min} \geq - \left(\frac{(c_1 + c_2\sqrt{2})^2}{4} + 2 + c_1 \right) \Theta_{\min} .$$

$$\Theta_{\min}(t) \geq c_0 \exp\left(-\left(\frac{(c_1 + c_2\sqrt{2})^2}{4} + 2 + c_1\right)t\right) > 0 \quad \text{on } [0, T) ,$$

completing the proof. \square

4.3. Uniform bounds on the height function. In this subsection, we show that the evolving surfaces stay in a compact region in M^3 throughout the flow. This is established via an estimate on the height function $u(x, t)$:

Theorem 4.5. *Suppose the mean curvature flow (1.1) has a solution on $[0, T)$, $0 < T \leq \infty$, then the height function $u(\cdot, t)$ (as defined in (4.6)) is uniformly bounded for $t \in [0, T)$.*

Proposition 3.4 plays a very important role in this key theorem. We use the hyperbolic properties of equidistant foliation, and Hopf's maximum principle to bound evolving surfaces of the mean curvature flow, i.e., hyperbolic geometry provides barrier surfaces for the mean curvature flow.

Proof. At each time $t \in [0, T)$, let $x(t) \in S(t)$ be the point such that

$$u_{\max}(t) \equiv \max_{x \in S(t)} u(x, t) = u(x(t), t) ,$$

and let $y(t) \in S(t)$ be the point such that

$$u_{\min}(t) \equiv \min_{y \in S(t)} u(y, t) = u(y(t), t) .$$

By the evolution equation (4.9) and the positivity of Θ along the flow (the Lemma (4.4)), we find the part of $S(t)$ with $H < 0$ will move along the positive direction

of \mathbf{n} while the part of $S(t)$ with $H > 0$ will move along the negative direction of \mathbf{n} , therefore we can assume that $u_{\max}(t)$ is increasing and $u_{\min}(t)$ is decreasing, for $t \geq t_0$, where $t_0 > 0$.

Our strategy is now clear: in the positive direction, at the furthest point on $S(t)$, the mean curvature is negative. We then apply the Proposition 3.4, that M^3 admits an equidistant foliation such that for far enough (at least r_0 from the reference surface), all fiber surfaces have positive mean curvatures. Fiber surfaces at r_0 and $-r_0$ then serve as barrier surfaces for the mean curvature flow (1.1) by Hopf's maximum principle.

We now follow the strategy: Since M^3 is almost Fuchsian, the parallel surfaces from the minimal surface Σ form the equidistant foliation, $\{\Sigma(r)\}_{r \in \mathbb{R}}$, of M^3 . Therefore, there exist $r_1 < r_2$ such that the surface $\Sigma(r_1)$ is tangent to $S(t)$ at the point $y(t)$, and the surface $\Sigma(r_2)$ is tangent to $S(t)$ at the point $x(t)$. It is easy to see that $u_{\max}(t) = r_2$ and $u_{\min}(t) = r_1$.

Let us assume the reference surface is the minimal surface Σ to apply some basic properties of principal curvatures for equidistant foliation $\{\Sigma(r)\}_{r \in \mathbb{R}}$, as in Proposition 3.4: from the formula (2.2), the principal curvatures of any point on $\Sigma(r)$ are determined by the principal curvatures of the corresponding point on Σ and r . In particular, if $\mu_1(x, r) < \mu_2(x, r)$ are principal curvatures of $(x, r) \in \Sigma(r)$, where $x \in \Sigma$, then we have, as in (2.2):

$$\mu_1(x, r) = \frac{\tanh(r) - \lambda(x)}{1 - \lambda(x) \tanh(r)}, \mu_2(x, r) = \frac{\tanh(r) + \lambda(x)}{1 + \lambda(x) \tanh(r)},$$

where $\pm\lambda(x)$ are the principal curvatures on the minimal surface Σ .

It is routine to verify that for fixed x , both $\mu_1(\cdot, r)$ and $\mu_2(\cdot, r)$ are increasing function of r , since $|\lambda(x)| < 1$. Let λ_0 be the maximum of the principal curvatures on the minimal surface Σ , then $0 < \lambda_0 < 1$.

Denote the constant $r_0 = \frac{1}{2} \log \frac{1+\lambda_0}{1-\lambda_0}$. The Proposition 2.4 says that, for any $r > r_0$, we have $\mu_1(x, r) > 0$ for all $x \in \Sigma$, hence all principal curvatures of the parallel surface $\Sigma(r)$ are positive. Similarly, for any $-r < -r_0$, all principal curvatures of the parallel surface $\Sigma(-r)$ are negative.

An easy modification to treat the general case when the reference surface $S \neq \Sigma$: since $|\lambda_j(S)| < 1$, we foliate the almost Fuchsian manifold M^3 by the equidistant foliation $\{S(r)\}_{r \in \mathbb{R}}$. Take now $\lambda_0 = \max\{|\lambda(S)|\}$ and $r_0 = \frac{1}{2} \log \frac{1+\lambda_0}{1-\lambda_0}$. Then the principal curvatures of $S(r)$ take forms of

$$\mu_j(x, r) = \frac{\tanh(r) + \lambda_j(x)}{1 + \lambda_j(x) \tanh(r)},$$

for $x \in S$ and $j = 1, 2$. Therefore again we find $-r_0 \leq r_1 < r_2 \leq r_0$.

We consider, at $F(x(t), t)$, $\Theta = \langle \mathbf{n}, \nu \rangle = 1$, then $0 \leq \frac{\partial u}{\partial t} = -H$, at $F(x(t), t)$. Therefore the mean curvature of $S(t)$ at the point $x(t)$ is non-positive. Since $\Sigma(r_2)$ is tangent to $S(t)$ at the positive side of $S(t)$, by Hopf's maximum principle (the Lemma 2.1), the mean curvature of $\Sigma(r_2)$ at $x(t)$ is no greater than that of the mean curvature of $S(t)$ at the intersection point $x(t)$. Therefore, there exists at least one point on the surface $\Sigma(r_2)$ with non-positive principal curvature. So the

evolving surfaces are uniformly bounded by two surfaces $S(-r_0)$ and $S(r_0)$, for all $t \in [t_0, T)$.

Combining with the bounds for $t \in [0, t_0]$, the height function is bounded in a compact region only depending on the reference surface S and initial surface S_0 . \square

As a corollary, since all evolving surfaces are staying within a compact region, we find:

Corollary 4.6. *There is a constant $c_3 > 0$, only depending on the initial surface S_0 and the minimal surface Σ , such that $c_3 \leq \Theta(\cdot, t) \leq 1$, for all $t \in [0, T)$.*

5. PROOF OF THEOREM 1.1: LONG TIME SOLUTION

We conclude the proof of the Theorem 1.1 in the section. To prevent singularity occurs in finite time, we need to derive uniform bounds for the second fundamental forms of the evolving surfaces. We use the uniqueness of the minimal surface in an almost Fuchsian manifold to show the mean curvature flow equation (1.1) converges to a unique limiting surface.

5.1. Uniform bounds on $|A|^2$ and H^2 . A crucial part of proving long time existence of the mean curvature flow equation (1.1) is to establish a priori bounds for the second fundamental forms on $S(t)$. In this subsection, we obtain such a bound. As a corollary, we obtain a uniform upper bound for the square of the mean curvature, $H^2(\cdot, t)$.

Theorem 5.1. *Suppose the mean curvature flow (1.1) has a solution for $t \in [0, T)$, then there is a constant $c_4 > 0$, only depending on S_0 , such that $|A|^2(\cdot, t) \leq c_4 < +\infty$, for all $t \in [0, T)$.*

Proof. The strategy is to add enough negative terms to (4.5). To this end, we derive the evolution equation satisfied by $\eta(\cdot, t) = \frac{1}{\Theta}$. This function is well-defined by the positivity of $\Theta(\cdot, t)$ (Lemma 4.4).

$$\begin{aligned} \frac{\partial}{\partial t} \eta &= -\eta^2 \frac{\partial}{\partial t} \Theta \\ &= \Delta \eta - 2\Theta |\nabla \eta|^2 - (|A|^2 - 2)\eta - \eta^2 J, \end{aligned}$$

where we denote $J = \mathbf{n}(H_{\mathbf{n}}) - H \langle \bar{\nabla}_{\nu} \mathbf{n}, \nu \rangle$.

We consider the function $f(\cdot, t) = |A|^2 \eta^4$, and we have

$$(5.1) \quad \frac{\partial}{\partial t} f = \eta^4 (\Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + 2) - 4H^2) + |A|^2 \frac{\partial}{\partial t} (\eta^4),$$

and we compute

$$\frac{\partial}{\partial t} (\eta^2) = 2\eta \frac{\partial}{\partial t} \eta = 2\eta \Delta \eta - 4|\nabla \eta|^2 - 2\eta^2(|A|^2 - 2) - 2\eta^3 J,$$

therefore we obtain:

$$(5.2) \quad L(\eta^2) = -6|\nabla \eta|^2 - 2\eta^2(|A|^2 - 2) - 2\eta^3 J,$$

where we introduce the operator $L = \frac{\partial}{\partial t} - \Delta$ to simplify our notation.

Therefore we have

$$\begin{aligned}
 \frac{\partial}{\partial t}(\eta^4) &= 2\eta^2(\Delta(\eta^2) - 6|\nabla\eta|^2 - 2\eta^2(|A|^2 - 2) - 2\eta^3 J) \\
 (5.3) \quad &= \Delta(\eta^4) - 20\eta^2|\nabla\eta|^2 - 4\eta^4(|A|^2 - 2) - 4\eta^5 J.
 \end{aligned}$$

Applying this into (5.1), we find

$$\begin{aligned}
 L(f) &= -2\eta^4|A|^4 - 2\eta^4|\nabla A|^2 + 4(3|A|^2 - H^2)\eta^4 \\
 &\quad - 2\nabla|A|^2 \cdot \nabla(\eta^4) - 20|A|^2\eta^2|\nabla\eta|^2 - 4|A|^2\eta^5 J,
 \end{aligned}$$

where the dominant term on the right is $-2\eta^4|A|^4 = -2\Theta^4 f^2$, for large $|A|^2$.

Now we assume $|A|^2$ is not uniformly bounded, then $|A|_{\max}^2(\cdot, t) \rightarrow \infty$ as $t \rightarrow T$. Since $f(\cdot, t) = |A|^2\eta^4 \geq |A|^2$, we have $f_{\max}(\cdot, t) \rightarrow \infty$ as $t \rightarrow T$. There then exists a $t_1 \in (0, T)$ such that when $t > t_1$, we have

$$\begin{aligned}
 \frac{d}{dt}f_{\max} &\leq -2\eta^4|A|^4 - 2\eta^4|\nabla A|^2 + 4(3|A|^2 - H^2)\eta^4 \\
 &\quad - 2\nabla|A|^2 \cdot \nabla(\eta^4) - 20|A|^2\eta^2|\nabla\eta|^2 - 4|A|^2\eta^5 J.
 \end{aligned}$$

From Corollary 4.6, we have $-2\eta^4|A|^4 = -2\Theta^4 f^2 \leq -2c_3^4 f^2$.

From the proof of Lemma 4.4, we estimate the term J :

$$|J| = |\mathbf{n}(H_{\mathbf{n}}) - H\langle \bar{\nabla}_{\nu}\mathbf{n}, \nu \rangle| \leq c_1\Theta^3 + (c_1|A| + c_2|H|)\Theta^2.$$

Since $c_3 \leq \Theta < 1$, for $t > t_1$, now we have

$$\frac{d}{dt}f_{\max} \leq -2c_3^4 f_{\max}^2 + \text{lower order terms}.$$

This is a contradiction since $\frac{d}{dt}f_{\max} > 0$. Therefore $f(\cdot, t) = |A|^2\eta^4$ is uniformly bounded, which in turn bounds $|A|^2$ from above. \square

As a corollary, we obtain the uniform bound for $H^2(\cdot, t)$:

Corollary 5.2. *Let $c_4 > 0$ be the constant as the upper bound of $|A|^2$ in Theorem 4.1, then $\sup_{S(t)}(H^2(\cdot, t)) \leq 2c_4$.*

Proof. This immediately follows from $|A|^2 - \frac{1}{2}H^2 \geq 0$. \square

5.2. Long time solution. We prove the mean curvature flow equation (1.1) admits long time solution, i.e., $T = +\infty$, in this subsection:

Theorem 5.3. *The maximal time of existence of the solution to the equation (1.1) is $T = +\infty$.*

We start with controlling the derivatives for the height function along the mean curvature flow:

Lemma 5.4. *If the mean curvature flow equation (1.1) has a solution on $[0, T)$, $0 < T \leq \infty$, then*

$$|\nabla^\ell u| \leq K_\ell < \infty,$$

for all $\ell = 1, 2, \dots$, where $\{K_\ell\}_{\ell=1}^\infty$ is the collection of constants depending only on the initial data and the maximal time T .

Proof. Since the evolving surfaces are graphical surfaces, we have, from [Hui86], $\Theta(\cdot, t) = 1/\sqrt{1 + |\nabla u|^2}$. From Corollary 4.6, there is a positive lower bound for $\Theta(\cdot, t)$, therefore $|\nabla u|$ is uniformly bounded from above by a constant depending only on the initial data and T .

We observe that equation (4.8):

$$\frac{\partial}{\partial t} u = -H\Theta = \Delta u - \operatorname{div}(\bar{\nabla} \ell)$$

is a single quasilinear parabolic equation for the height function $u(\cdot, t)$, while $u(\cdot, t)$ is uniformly bounded by the Theorem 4.5. This enables us to apply the standard regularity results in quasilinear second order parabolic equations ([Fri64], [Lie96]) to bound all derivatives of u . \square

Using Lemma 5.4 and the relation $\Theta(\cdot, t) = 1/\sqrt{1 + |\nabla u|^2}$, we obtain upper bounds for the derivatives of the gradient function $\Theta(\cdot, t)$:

Corollary 5.5. *There exist constants $0 < K'_\ell < \infty$ depending only on S_0 and T such that*

$$|\nabla^\ell \Theta|^2 \leq K'_\ell$$

on $S(t)$ for $0 \leq t < T$, and $\ell = 1, 2, \dots$

Having bounded the height function u , the gradient function Θ , and their derivatives, we obtain the estimates for the derivatives of the second fundamental form:

Proposition 5.6. *For each $n \geq 1$, there is a constant $c_5(n, S_0) > 0$ such that $|\nabla^n A|^2 \leq c_5(n, S_0)$ uniformly on $S(t)$, for $t \in [0, T)$.*

Proof. The basic evolution equation for $|\nabla^n A|^2$ is ([Ham82], [Hui84]):

$$L(|\nabla^n A|^2) = -2|\nabla^{n+1} A|^2 + K''_{i,j,k,n} \sum_{i+j+k=n} \nabla_i A * \nabla_j A * \nabla_k A * \nabla_n A,$$

where $L = \frac{\partial}{\partial t} - \Delta$, and $K''_{i,j,k,n}$ is a constant depending on S_0 , and natural numbers i, j, k , and n with $i + j + k = n$. The upper bounds for all derivatives of A follow from an induction on n . \square

With all the pieces in place, we conclude this subsection with:

proof of Theorem 5.3. By the Theorem 4.5, the height function $u(\cdot, t)$ is uniformly bounded, therefore, the surfaces $\{S(t)\}_{t \in [0, T]}$ stay in a compact smooth region in M^3 . Applying the Theorem 5.1, Corollary 5.2, Corollary 5.5 and Proposition 5.6, the sequence $\{S(t)\}$ converges to a limiting smooth surface S_T as $t \rightarrow T$, which is again, a graph. Now we use S_T as our initial surface in the equation (1.1) to extend the flow beyond T , by the existence of short-time solutions result for the new parabolic equation (Theorem 2.3). \square

5.3. Convergence and uniqueness. In this subsection, we conclude the proof of our Theorem 1.1 by showing the convergence and uniqueness of the limiting surface to the equation (1.1). We first establish the mean curvature estimate for the limiting surface.

Theorem 5.7. *Let $H(\cdot, t)$ be the mean curvature functions of the evolving surfaces $S(t)$, then $\sup_{S(t)} |H(\cdot, t)| \rightarrow 0$ as $t \rightarrow \infty$.*

Proof. We consider the function $h(t) = \int_{S(t)} H^2 d\mu$. From the formula (4.3), we have

$$(5.4) \quad \frac{d}{dt} |S(t)| = -h(t) \leq 0,$$

where $|S(t)|$ is the surface area of $S(t)$.

We integrate both sides of (5.4), since $T = \infty$, to find

$$\int_0^\infty h(t) dt = |S_0| - |S(\infty)| \leq |S_0|.$$

Because the initial surface S_0 is a closed incompressible surface, we know that $|S_0|$ is finite, hence the function $h(t)$ is uniformly bounded.

We now compute the t -derivative of this function $h(t)$:

$$\begin{aligned} \frac{d}{dt} h &= \frac{d}{dt} \int_{S(t)} H^2 d\mu \\ &= \int_{S(t)} 2HH_t - H^4 d\mu \\ &= \int_{S(t)} -2|\nabla H|^2 + 2H^2(|A|^2 - 2) - H^4 d\mu. \end{aligned}$$

Note that we also have uniform bound for $|\nabla H(\cdot, t)|$, which follows from (4.2), Theorem 4.5, and Corollary 5.5, or Proposition 4.6. Applying this and upper bounds on $|A|^2$ (Theorem 5.1), we find that the term $|\frac{d}{dt} h(t)|$ is also uniformly bounded in t . Now the function $h(t)$ is bounded in t , with bounded derivative in t , therefore it must tend to zero as $t \rightarrow \infty$.

Now a standard interpolation argument, with the uniform bounds on H and $|\nabla H(\cdot, t)|$, shows that $\sup_{S(t)} |H| \rightarrow 0$ as $t \rightarrow \infty$. \square

Proof. (of Theorem 1.1) Part (i) is proved by Theorem 5.3, and part (ii) is implied by Lemma 4.4. We are left to show convergence and properties of the limiting surface.

From Theorem 5.7, every sequence $S(t)$ has a subsequence $S(t_i)$ converging smoothly to some stationary asymptotic limiting surface, say $S_i(\infty)$. It is clear that this limiting surface is closed and incompressible which satisfies that its mean curvature is zero, hence minimal. However, the ambient space is almost Fuchsian, i.e., M^3 only admits one unique incompressible minimal surface, which is Σ .

The convergence is exponential. This can be seen as follows: for t large enough, since evolving surfaces are closed, and the minimal surface has principal curvatures of absolute values less than one, we have $|A|^2 \leq 2 - \delta$, for some constant $\delta > 0$.

Then we find, in the proof of the Theorem 5.7: $\frac{d}{dt}h(t) \leq -2\delta h(t)$, which forces the exponential convergence. \square

6. APPLICATIONS

In this section, we apply our method to more general cases. Applications include other class of hyperbolic three-manifolds, as well as more geometrical properties of the mean curvature flow if better initial data is chosen.

6.1. Nearly Fuchsian manifolds. In this subsection, we consider a slightly larger class of quasi-Fuchsian manifolds than almost Fuchsian manifolds, i.e.,

Definition 6.1. *A quasi-Fuchsian manifold N is called **nearly Fuchsian** if it contains a closed incompressible surface (not necessarily minimal) of principal curvatures within the unit interval $(-1, 1)$.*

It is not known if any nearly Fuchsian manifold is in fact almost Fuchsian.

From the proof of Theorem 1.1, after a modification of the argument, it is easy to see that parts (i) and (ii) of Theorem 1.1 hold in this class:

Theorem 6.2. *Let N be a nearly Fuchsian three-manifold which contains a closed incompressible surface S whose principal curvatures lying in $(-1, 1)$. If a smooth closed surface S_0 in N is a graph over S : there is a constant $c_0 > 0$ such that $\langle \mathbf{n}, \nu_0 \rangle \geq c_0$, where \mathbf{n} and ν_0 are as in Theorem 1.1. Then:*

- (i) *the mean curvature flow equation (1.1) with initial surface $S(0) = S_0$ has a long time solution;*
- (ii) *the evolving surfaces $\{S(t)\}_{t \in \mathbb{R}}$ stay smooth and remain as graphs over S for all time;*
- (iii) *For each sequence $\{t\}$, there is a subsequence $\{t'\}$ such that evolving surfaces $\{S(t')\}_{t' \rightarrow \infty}$ converge smoothly to a minimal surface $S(\infty)$, which is embedded.*

The flow limit $S(\infty)$ is unique for a given initial surface ([Sim83]). However, we note that in comparison with Theorem 1.1, without assuming N is almost Fuchsian, it is possible that N might contain several minimal surfaces since different initial surfaces may be deformed into different minimal surfaces. In [Wan10], certain quasi-Fuchsian manifolds contain several minimal surfaces.

We note that Theorem 1.3 also holds for nearly Fuchsian three-manifolds, i.e., if M^3 is nearly Fuchsian, and S is a closed incompressible surface with maximal absolute principal curvature $\lambda_0 = \max\{|\lambda(S)|\} < 1$, then the Hausdorff dimension of the limit set associate to M^3 is less than $1 + \lambda_0^2$.

6.2. Mean-convexity. Theorems 1.1 and 6.2 indicate that the mean curvature flow (1.1) is quite insensitive about the initial data. As an application, we consider more specific initial surfaces, i.e., mean-convex surfaces:

Theorem 6.3. *Let N be a nearly Fuchsian three-manifold with a closed incompressible surface S whose principal curvatures lying in $(-1, 1)$. If a smooth closed*

surface S_0 in N is a graph over S such that S_0 has positive mean curvature everywhere. Let $\{S(t)\}_{t \in \mathbb{R}}$ be evolving surfaces for the mean curvature flow equation (1.1) with initial surface $S(0) = S_0$, then:

- (i) $H(S(t)) > 0$ for all $t \in \mathbb{R}$;
- (ii) the height function $u(\cdot, t)$ is decreasing in t .

Proof. Part (ii) is an easy consequence of part (i) and the equation (4.8). We only show mean curvature flow preserves the positivity of mean curvatures.

The existence of such surfaces of positive mean curvature everywhere is obvious: by Proposition 3.4 and proof of Theorem 4.5, there is a $r_0 > 0$, only depending on S , such that each parallel $\{S(r)\}_{r > r_0}$ to S has positive principal curvatures everywhere. We might just take any such $S(r)$ as the initial surface S_0 for the mean curvature flow (1.1).

Let $H_{\min}(\cdot, t)$ be the minimal mean curvature of the surface $S(t)$. Recall from the evolution equation (4.4) for $H(\cdot, t)$, i.e.,

$$H_t = \Delta H + H(|A|^2 - 2).$$

And our initial surface satisfies $H(\cdot, 0) > 0$ by assumption. Let t_0 be the first time that $H_{\min}(t_0) = 0$, then at t_0 and at where H_{\min} occurs, we have

$$\frac{d}{dt}|_{t=t_0} H_{\min} \geq H_{\min}(t_0)(|A|^2 - 2) = 0.$$

Here we also note the fact that the surface $S(t_0)$ is closed. This now implies that $H \geq 0$ for all time so long the solution to the mean curvature flow exists. This means we can refine the estimate to, at t_0 and H_{\min} ,

$$\frac{d}{dt}|_{t=t_0} H_{\min} \geq H_{\min}(t_0)(|A|^2 - 2) \geq -2H_{\min},$$

which we arrive at the conclusion $H_{\min}(t) \geq H_{\min}(0)e^{-2t} > 0$. \square

Similar results hold for mean curvature flow with initial surface S'_0 of negative mean curvature everywhere, with reversed inequality for the height function.

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